Minimum-Phase Behavior of Random Media

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***Abstract*—We give two different sufficient conditions for the transfer function of two-mode random media to be minimum phase. The second of these results states that the signal transfer function will be minimum phase if the spurious mode is dissipated faster than it is coupled from the signal mode, in a certain sense. Two illustrative examples are given. These two conditions cannot be greatly improved.**

I. INTRODUCTION

TRANSMISSION MEDIA with coupled spurious modes have been of interest in waveguide and fiber optic systems. Here we are concerned with a two-mode system, in which signal and spurious modes propagate in the forward direction. Previous studies [1]–[3] have given some exact, approximate, and statistical properties of the signal-mode transfer function of such a system.

It is of interest to know whether such a transfer function is minimum phase. Recall [4] that a minimum-phase transfer function has no zeros in the right-half of the complex frequency plane; and hence the phase may be uniquely determined from the attenuation (log of the magnitude) of the transfer function. We present here sufficient conditions for the signal transfer function to be minimum phase, and show that they cannot be greatly improved.

Consider the coupled line equations

$$I_0'(z) = -\Gamma_0 I_0(z) + jc(z) I_1(z)$$
$$I_1'(z) = jc(z) I_0(z) - \Gamma_1 I_1(z) \tag{1}$$

describing a system of two coupled modes traveling in the $+z$ direction. Γ_0 and Γ_1 are the complex propagation constants, with real and imaginary parts

$$\Gamma_0 \equiv \alpha_0 + j\beta_0, \qquad \Gamma_1 \equiv \alpha_1 + j\beta_1 \tag{2}$$

and $'$ denotes differentiation with respect to z. $I_0(z)$ and $I_1(z)$ are coupled wave amplitudes representing signal and spurious modes, respectively, each having time dependence $\exp(j2\pi ft)$. $c(z)$ is a real coupling coefficient having arbitrary functional dependence on distance z; $c(z)$ is taken as a random process with known statistics in some problems.

We assume initial conditions

$$I_0(0) = 1, \qquad I_1(0) = 0. \tag{3}$$

Thus, a unit signal is injected in the desired mode at $z = 0$; the output $I_0(z)$ is then the complex-signal transfer function for length z of guide.

The following normalization is convenient [1]:

$$I_0(z) \equiv \exp(-\Gamma_0 z) \cdot G_0(z)$$
$$I_1(z) \equiv \exp(-\Gamma_1 z) \cdot G_1(z) \tag{4}$$

$$\Delta\Gamma \equiv \Gamma_0 - \Gamma_1 = \Delta\alpha + j\Delta\beta \tag{5}$$

$$\Delta\alpha = \alpha_0 - \alpha_1$$
$$\Delta\beta = \beta_0 - \beta_1. \tag{6}$$

Then (1) becomes

$$G_0'(z) = jc(z) \exp(\Delta\Gamma z) \cdot G_1(z)$$
$$G_1'(z) = jc(z) \exp(-\Delta\Gamma z) \cdot G_0(z) \tag{7}$$

governing the normalized transfer functions G_0 and G_1. The initial conditions (3) become

$$G_0(0) = 1, \qquad G_1(0) = 0. \tag{8}$$

Now let $G_0(\Delta\Gamma)$ be the solution to (7) and (8) for some fixed guide length z and coupling function $c(z)$. Then the normalized-signal transfer function for a guide with particular values of attenuation and phase constants is found by substituting (6) for the real and imaginary parts of the complex parameter $\Delta\Gamma$ of (5).

The physical applications of these equations have been discussed in several places, in particular in [1, sec. I], which gives earlier pertinent references. The following facts [1, secs. II and III and appendix I], are of direct interest here.

1) The signal mode is assumed to have lower heat loss and greater group velocity than the spurious mode

$$\Delta\alpha = \alpha_0 - \alpha_1 \leq 0 \tag{9}$$

$$(d/df)\Delta\beta < 0. \tag{10}$$

2) Over narrow bands of interest we neglect the frequency dependence of $\Delta\alpha$ and $c(z)$, and assume $\Delta\beta$ varies approximately linearly with frequency f (with negative slope).

Item 2) suggests the substitution

$$-\Delta\beta = \lambda \tag{11}$$

where λ is normalized angular frequency, since

$$\lambda \approx \text{constant} \cdot 2\pi f \tag{12}$$

over a suitably narrow band. Introduce the complex frequency

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$$s \equiv \sigma + j\lambda \tag{13}$$

as with the LaPlace transform. Then from [1, secs. IV and V and appendix II]

$$G_0(\Delta\alpha - s) = G_0(\Delta\alpha - \sigma - j\lambda) \tag{14}$$

gives the behavior of the transfer function G_0 throughout the complex frequency plane, where $G_0(\Delta\Gamma)$ is the solution to (7) and (8) for some fixed guide length.

Therefore let us relate the two complex $\Delta\Gamma$ and s planes by

$$\Delta\Gamma = \Delta\alpha - s \tag{15}$$

where $\Delta\alpha$ is the particular fixed value of differential attenuation under consideration. The right-half s plane,

$$\sigma > 0 \tag{16a}$$

corresponds to the region

$$\mathrm{Re}\,\Delta\Gamma < \Delta\alpha = -|\Delta\alpha| \tag{16b}$$

in the $\Delta\Gamma$ plane. The imaginary axis in the s plane

$$\sigma = 0 \quad s = j\lambda \tag{17a}$$

corresponds to the vertical line

$$\Delta\Gamma = \Delta\alpha - j\lambda = \Delta\alpha + j\Delta\beta \tag{17b}$$

in the left-half $\Delta\Gamma$ plane, and G_0 evaluated at points along (17b) gives the transfer function for sinusoidal inputs, the only values of direct physical interest.

II. RESULTS

A number of general properties of G_0 were given in [1, sec. IV] for arbitrary $c(x)$, valid without perturbation or any other approximations. Of particular interest here, G_0 is analytic for all finite $\Delta\Gamma$ (and hence for all finite s); i.e., G_0 has no poles or other singularities anywhere in the finite $\Delta\Gamma$ or s planes. However, [1] contained no information about zeros of G_0. Such information is contained in the following results.

1) If

$$\int_0^z |c(x)|\,dx < \cosh^{-1} 2 \approx 1.317 \tag{18}$$

then all zeros of G_0 lie in the region of the s plane

$$\sigma \equiv \mathrm{Re}\,s < \Delta\alpha \leq 0. \tag{19}$$

G_0 is consequently minimum phase, since all zeros lie in the left-half s plane.

2) If

$$\int_0^\zeta |c(x)| \exp[\Delta\alpha(\zeta - x)]dx \leq \left(\frac{\sqrt{2}-1}{2}\right)^{1/2} \approx 0.455, \quad \text{for } 0 \leq \zeta \leq z,\ \Delta\alpha \leq 0 \tag{20}$$

then G_0 is minimum phase, i.e., all zeros of G_0 lie in the left-half s plane

$$\sigma \equiv \mathrm{Re}\,s < 0. \tag{21}$$

III. SERIES SOLUTIONS AND MINIMUM PHASE

A general series solution for G_0 of (7) and (8), analytic for all finite $\Delta\Gamma$, is given in [1] and [2]. From [1, appendix II]

$$G_0(\Delta\Gamma) = 1 + \sum_{n=1}^{\infty} (-1)^n G_{0(n)}(\Delta\Gamma) \tag{22}$$

where the terms are bounded by

$$|G_{0(n)}(\Delta\Gamma)| \leq \left[\int_0^z |c(x)|\,dx\right]^{2n} \Big/ (2n)!, \qquad \mathrm{Re}\,\Delta\Gamma \leq 0. \tag{23}$$

The precise form for $G_{0(n)}(\Delta\Gamma)$, and bounds for $\mathrm{Re}\,\Delta\Gamma > 0$, are given in [1], but are not of interest here. The $G_{0(n)}$ of (22) are in general complex. Then

$$G_0(\Delta\Gamma) \neq 0$$

if

$$\left|\sum_{n=1}^{\infty} (-1)^n G_{0(n)}(\Delta\Gamma)\right| < 1. \tag{24}$$

Using (23)

$$\left|\sum_{n=1}^{\infty} (-1)^n G_{0(n)}(\Delta\Gamma)\right| \leq \sum_{n=1}^{\infty} |G_{0(n)}(\Delta\Gamma)| \leq \sum_{n=1}^{\infty} \cdot\left[\int_0^z |c(x)|\,dx\right]^{2n} \Big/ (2n)! = \cosh\left[\int_0^z |c(x)|\,dx\right] - 1, \qquad \mathrm{Re}\,\Delta\Gamma \leq 0. \tag{25}$$

Therefore $G_0(\Delta\Gamma)$ has no zeros in the left-half $\Delta\Gamma$ plane if

$$\cosh\left[\int_0^z |c(x)|\,dx\right] < 2 \tag{26}$$

or

$$\int_0^z |c(x)|\,dx < \cosh^{-1} 2 \approx 1.317. \tag{27}$$

By (16), the condition of (27) and (18) excludes zeros from the right-half s plane, guaranteeing that G_0 is minimum phase. Moreover, zeros are excluded from a strip extending from $\mathrm{Re}\,s = 0$ to $\mathrm{Re}\,s = \Delta\alpha \leq 0$ in the left-half s plane, as stated in (19).

Define the complex signal loss as [2]

$$\Lambda(\Delta\Gamma) = -\ln G_0(\Delta\Gamma). \tag{28}$$

A series solution for Λ was given in [2]. In the region

$$\mathrm{Re}\,\Delta\Gamma \leq \Delta\alpha \leq 0 \tag{29}$$

this series converges, and hence Λ is analytic, if the condition

$$\int_0^\zeta |c(x)| \exp [\Delta\alpha(\zeta - x)]dx \leq \left(\frac{\sqrt{2}-1}{2}\right)^{1/2} \approx 0.455,$$

$$\text{for} \quad 0 \leq \zeta \leq z, \Delta\alpha \leq 0 \quad (30)$$

given in (20), is satisfied. Since poles of Λ correspond to zeros and poles of G_0, and since G_0 has no finite poles, absence of poles for Λ implies absence of zeros for G_0. From (29) and (16), the condition of (20) or (30) excludes zeros from the right-half s plane, as stated in (21), guarantees G_0 to be minimum phase.

IV. EXAMPLES

Consider first two equal discrete-mode converters separated by a length z of ideal guide

$$c(x) = C[\delta(x) + \delta(x - z)]. \quad (31)$$

The output quantities of (1) for the inputs (3) are given by [1, sec. VI] as

$$\begin{bmatrix} I_0(z+) \\ I_1(z+) \end{bmatrix} = \begin{bmatrix} \cos C & j \sin C \\ j \sin C & \cos C \end{bmatrix} \begin{bmatrix} \exp(-\Gamma_0 z) & 0 \\ 0 & \exp(-\Gamma_1 z) \end{bmatrix} \cdot \begin{bmatrix} \cos C & j \sin C \\ j \sin C & \cos C \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \quad (32)$$

Then

$$I_0(z+) = \exp(-\Gamma_0 z) \cos^2 C - \exp(-\Gamma_1 z) \sin^2 C. \quad (33)$$

From (4), for length $z+$

$$G_0(\Delta\Gamma) = \cos^2 C - \exp(\Delta\Gamma z) \sin^2 C. \quad (34)$$

From (15)

$$G_0(\Delta\alpha - s) = \cos^2 C - \exp[(\Delta\alpha - s)z] \sin^2 C. \quad (35)$$

Clearly, there are no finite poles. Zeros occur for

$$\exp[(s - \Delta\alpha)z] = \tan^2 C. \quad (36)$$

Substituting (13), zeros occur at

$$\lambda z = \text{integer} \cdot 2\pi \quad (37)$$

and

$$\exp[(\sigma - \Delta\alpha)z] = \tan^2 C. \quad (38)$$

Recalling (9), zeros are confined to the region

$$\sigma < \Delta\alpha \leq 0 \quad (39)$$

in the present special case if

$$\tan^2 C < 1, \qquad |C| < (\pi/4) \approx 0.785. \quad (40)$$

If $<$ is replaced by $=$ in (40), zeros appear along the vertical line $\sigma = \Delta\alpha \leq 0$ in the s plane; therefore, the numerical factor on the right sides of (40) cannot be increased if zeros are restricted by (39) for $c(x)$ of (31). The general condition of (18) applied to the present case requires

$$|C| < \frac{\cosh^{-1} 2}{2} \approx 0.658 \quad (41)$$

this general result is consequently a little conservative when applied to the present special case.

Consider, finally, constant coupling between signal and spurious modes

$$c(x) = c_0. \quad (42)$$

In this case I_0 of (1) is given by [3, sec. 2.3.3] as

$$I_0(z) = \exp\left(-\frac{\Gamma_0 + \Gamma_1}{2} z\right) \cdot \frac{1}{K_+ - K_-} \cdot \{-K_- \exp[(\Delta\Gamma/2)z\sqrt{\ }] + K_+ \exp[-(\Delta\Gamma/2)z\sqrt{\ }]\} \quad (43)$$

where

$$K_\pm = -j \frac{1 \pm \sqrt{\ }}{2c_0/\Delta\Gamma}, \qquad K_+K_- = -1,$$

$$K_+ - K_- = -j2 \frac{\sqrt{\ }}{2c_0/\Delta\Gamma} \quad (44)$$

$$\sqrt{\ } = \left[1 - \left(\frac{2c_0}{\Delta\Gamma}\right)^2\right]^{1/2}. \quad (45)$$

Consequently, from (4), for length z

$$G_0(\Delta\Gamma) = \exp[(\Delta\Gamma/2)z] \cdot \frac{1}{K_+ - K_-} \cdot \{-K_- \exp[(\Delta\Gamma/2)z\sqrt{\ }] + K_+ \exp[-(\Delta\Gamma/2)z\sqrt{\ }]\}. \quad (46)$$

Now G_0 appears to have branch points at $\Delta\Gamma = \pm 2c_0$. However, reversing the sign of $\sqrt{\ }$ in (46) leaves this expression unaltered, so there are in fact no branch points. This is in accord with the general result quoted at the beginning of Section II, that G_0 is analytic for all finite $\Delta\Gamma$ for arbitrary coupling $c(z)$ [1].

The zeros of the normalized transfer function G_0 are given from (44)–(46) by solving

$$\frac{(1 + \sqrt{\ }) \exp\left(-\frac{\Delta\Gamma}{2} z\sqrt{\ }\right) - (1 - \sqrt{\ }) \exp\left(\frac{\Delta\Gamma}{2} z\sqrt{\ }\right)}{2\sqrt{\ }} = 0. \quad (47)$$

We first explore the exceptional points given by the zeros of the denominator

$$\Delta\Gamma = \pm 2c_0. \tag{48}$$

Taking limits in (46)

$$G_0(\Delta\Gamma = \pm 2c_0) = [1 - (\Delta\Gamma/2)z] \exp [(\Delta\Gamma/2)z]. \tag{49}$$

Since $z > 0$, G_0 cannot have a zero for $\Delta\Gamma = -2 \mid c_0 \mid$. G_0 has a zero for $\Delta\Gamma = +2 \mid c_0 \mid$ if and only if $\mid c_0 \mid z = 1$. The remaining zeros of G_0 are given by setting the numerator of (47) equal to 0

$$\frac{\{1 + [1 - (2c_0/\Delta\Gamma)^2]^{1/2}\}^2}{(2c_0/\Delta\Gamma)^2} \exp \left\{ -\Delta\Gamma z \left[1 - \left(\frac{2c_0}{\Delta\Gamma} \right)^2 \right]^{1/2} \right\} = 1, \qquad \Delta\Gamma \neq \pm 2 \mid c_0 \mid. \tag{50}$$

We are free to replace c_0 by $\mid c_0 \mid$ throughout (50).

Summarizing, for constant coupling (42) the zeros of the transfer function are given by solving

$$\left\{ \frac{\Delta\Gamma}{2 \mid c_0 \mid} + \left[\left(\frac{\Delta\Gamma}{2 \mid c_0 \mid} \right)^2 - 1 \right]^{1/2} \right\}^2 \exp \left\{ -2 \mid c_0 \mid z \cdot \left[\left(\frac{\Delta\Gamma}{2 \mid c_0 \mid} \right)^2 - 1 \right]^{1/2} \right\} = 1, \qquad \Delta\Gamma \neq \pm 2 \mid c_0 \mid \tag{51}$$

for $\Delta\Gamma$, and additionally

$$\Delta\Gamma = 2 \mid c_0 \mid \quad \text{if and only if} \quad \mid c_0 \mid z = 1. \tag{52}$$

Recalling (15), the transformation to the complex frequency plane is given by

$$\Delta\Gamma = \Delta\alpha - s. \tag{53}$$

Investigation of (51)–(53) in the Appendix yields the following two results on transfer-function zeros for constant coupling (42):

<u>$c(x) = c_0$</u>

1) If

$$\mid c_0 \mid z < (\pi/2) \approx 1.571 \tag{54}$$

then all zeros of G_0 lie in the region of the s plane

$$\sigma \equiv \operatorname{Re} s < \Delta\alpha \leq 0. \tag{55}$$

G_0 is consequently minimum phase, since all zeros lie in the left-half s plane.

2) If

$$\mid c_0 \mid \leq 0.5 \mid \Delta\alpha \mid \tag{56}$$

then G_0 is minimum phase, i.e., all zeros of G_0 lie in the left-half s plane

$$\sigma \equiv \operatorname{Re} s < 0. \tag{57}$$

The numerical factors on the right sides of (54) and (56) cannot be increased for constant coupling if zeros are restricted by (55) and (57), respectively; this follows from the special case $-\Delta\beta = \lambda = 0$, investigated at the end of the Appendix. The general results of (18) and (19) and (20) and (21) applied to the present case replace the numerical factors 1.571 of (54) by 1.317, 0.5 of (56) by 0.455; these general results are again a little conservative when applied to the present special case.

V. DISCUSSION

The conditions of (18) and (20) are sufficient, for arbitrary coupling $c(x)$, to exclude transfer-function zeros from the right half of the complex frequency plane, and so guarantee the transfer function to be minimum phase. When $\mid \Delta\alpha \mid > 0$, the condition (18) is stronger than required for minimum-phase behavior, excluding transfer-function zeros from a vertical strip in the left-half plane as well.

The condition of (18) permits larger coupling $c(x)$ than that of (20) for $\Delta\alpha = 0$; for large enough $\mid \Delta\alpha \mid z$ and most reasonable $c(x)$, the reverse is true. If a given $c(x)$ satisfies both constraints (18) and (20), (19) provides a stronger constraint on the transfer-function zeros than (21). In a rough sense, (20) and (21) state that the transfer function is minimum phase if the spurious mode is dissipated faster than it is coupled from the signal mode.

The examples of Section IV show that little improvement is possible in the general results of Section II.

APPENDIX

For convenience introduce the notation

$$W \equiv \frac{\Delta\Gamma}{2 \mid c_0 \mid}. \tag{58}$$

Then (51) and (52) become, respectively,

$$[W + (W^2 - 1)^{1/2}]^2 \exp [-2 \mid c_0 \mid z(W^2 - 1)^{1/2}] = 1, \qquad W \neq \pm 1 \tag{59}$$

$$W = 1 \quad \text{if and only if} \quad \mid c_0 \mid z = 1. \tag{60}$$

From (15) or (53) and (13)

$$W = -\frac{\mid \Delta\alpha \mid}{2 \mid c_0 \mid} - \frac{\sigma}{2 \mid c_0 \mid} - j \frac{\lambda}{2 \mid c_0 \mid}. \tag{61}$$

Then we have the following correspondences:

$$\sigma > - \mid \Delta\alpha \mid \Leftrightarrow \operatorname{Re} W < 0$$

$$\sigma > 0 \Leftrightarrow \operatorname{Re} W < -\frac{\mid \Delta\alpha \mid}{2 \mid c_0 \mid}. \tag{62}$$

To investigate the solutions of (59) we introduce for convenience the auxiliary complex planes

$$X \equiv (W^2 - 1)^{1/2}$$

$$Y \equiv W + (W^2 - 1)^{1/2}. \tag{63}$$

X and Y are double-valued. We render them single-valued for our present purposes by making a branch cut from -1 to 1 in the W plane, choosing the branches by requiring that

$$X \sim W \quad \text{and} \quad Y \sim 2W \quad \text{as} \quad |W| \to \infty. \tag{64}$$

The arbitrariness of this choice is unimportant for what follows.[1] Then (59) becomes

$$Y = \pm \exp(|c_0| zX), \qquad W \neq \pm 1. \tag{65}$$

Fig. 1 shows corresponding contours in the W, X, and Y planes. The shaded regions correspond, respectively, to: 1) $\operatorname{Re} W < 0$, $\operatorname{Im} W \neq 0$, and $-\infty < W < -1$; 2) $\operatorname{Re} X < 0$; 3) $|Y| > 1$, $\operatorname{Re} Y < 0$. It is clear that (65) has no solution here. Consequently, (59) has no solutions in the left-half W plane with the possible exception of the portion of the real axis $-1 < W < 0$.

Now (65) and hence (59) clearly have no solutions when $\operatorname{Re} W \leq -1$. From (62), if

$$\frac{|\Delta\alpha|}{2|c_0|} \geq 1, \qquad |c_0| \leq 0.5 |\Delta\alpha| \tag{66}$$

the transfer-function zeros must lie in the left-half s plane, as stated in (56) and (57).

Next let us see what additional conditions are necessary to exclude solutions to (65), and hence (59), from the portion of the negative real axis $-1 < W \leq 0$ and from the imaginary axis $\operatorname{Re} W = 0$, thereby confining transfer-function zeros to $\sigma < -|\Delta\alpha|$ by (62). The region under investigation corresponds to the imaginary X axis, excluding the origin; therefore, set

$$X = jx, \qquad x \neq 0. \tag{67}$$

Then (65) becomes

$$\pm \exp(j|c_0|zx) = \begin{cases} \exp\{j[\pi - \sin^{-1} x]\}, & 0 < |x| \leq 1 \\ jx[1 + (1 - 1/x^2)^{1/2}], & 1 \leq |x|. \end{cases} \tag{68}$$

Solutions to (68) are forbidden if

$$|c_0| z < (\pi/2) \tag{69}$$

thus yielding (54) and (55).

Finally, let us investigate the boundary cases to demonstrate that our bounds (54) and (56) on the coupling coefficient cannot be relaxed. For constant coupling, consider the special case

$$-\Delta\beta = \lambda = 0. \tag{70}$$

First, note that (68) has a solution if

$$|c_0| z = (\pi/2) \tag{71}$$

at

$$x = 1 \tag{72}$$

by (67) and (63) this corresponds to

[1] Because (59) is single-valued.

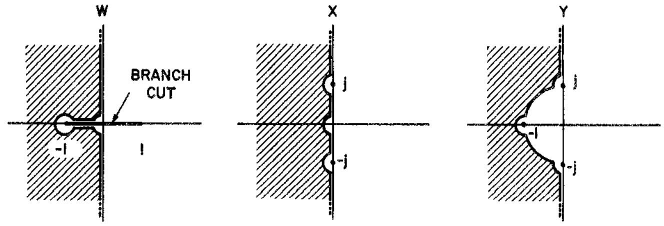

Fig. 1. Contours in three complex planes.

$$W = 0 \tag{73}$$

or by (61)

$$\sigma - \Delta\alpha = 0. \tag{74}$$

Hence replacing $<$ by $=$ in (54) results in an s-plane zero at

$$\sigma = \Delta\alpha \quad \lambda = 0 \tag{75}$$

showing that the numerical factor on the right side of (54) cannot be increased with zeros restricted by (55). Stated differently, evaluate (46) at the origin of the $\Delta\Gamma$ plane, to yield

$$G_0(\Delta\Gamma = 0) = \cos c_0 z. \tag{76}$$

Clearly, this is zero if $|c_0| z = \pi/2$, and from (13) and (15) or (53) the corresponding zero in the s plane is as given previously in (75). For a second example, let

$$|c_0| z > (\pi/2) \tag{77}$$

then (68) has a solution for

$$0 < x < 1. \tag{78}$$

By (63) and (67)

$$W = -(1 - x^2)^{1/2} \tag{79}$$

(78) yields

$$-1 < W < 0. \tag{80}$$

From (61) and (79)

$$\frac{\sigma}{2|c_0|} = (1 - x^2)^{1/2} - \frac{|\Delta\alpha|}{2|c_0|}. \tag{81}$$

Now we can make $|c_0| z$ large enough to make $(1 - x^2)^{1/2}$ as close to 1 as we please. Hence if

$$\frac{|\Delta\alpha|}{2|c_0|} < 1 \tag{82}$$

we can make $|c_0| z$ large enough to render

$$\frac{\sigma}{2|c_0|} > 0 \tag{83}$$

by (82). Hence replacing $\leq$ by $>$ in (56) can result in an s-plane zero in the right-half plane

$$\sigma > 0 \quad \lambda = 0 \tag{84}$$

if $|c_0| z$ is sufficiently large. This demonstrates that the numerical factor on the right side of (56) cannot be increased with zeros restricted by (57).

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REFERENCES

[1] H. E. Rowe and D. T. Young, "Transmission distortion in multimode random waveguides," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-20, pp. 349–365, June 1972.
[2] H. E. Rowe, "Approximate solutions for the coupled line equations," *Bell Syst. Tech. J.*, vol. 41, pp. 1011–1029, May 1962.
[3] H. E. Rowe and W. D. Warters, "Transmission in multimode waveguide with random imperfections," *Bell Syst. Tech. J.*, vol. 41, pp. 1031–1170, May 1962.
[4] A. Papoulis, *The Fourier Integral and Its Applications.* New York: McGraw-Hill, 1962, pp. 204–212.

Statistical Coupled Equations in Lossless Optical Fibers

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Abstract—The problem of deriving sets of statistical coupled equations for the second and fourth moments of the mode amplitudes in a fiber with mode coupling is considered, starting from the deterministic coupled wave equations describing an electromagnetic field propagating in a lossless fiber. Our results extend the work of Marcuse, and, in particular, allow one to deduce sets of equations for quantities which describe the cross correlation between different modes. Furthermore, we obtain new results regarding the variances and cross correlations of the power in the modes (fourth-order amplitude statistics).

I. INTRODUCTION

AN electromagnetic wave propagating in an optical fiber can be described by means of a set of coupled differential equations for the amplitudes of the modes supported by the guide. The coupling terms are, in particular, associated with the deviations of the fiber from the ideal structure pertaining to a regular geometrical form and refractive index distribution. In many situations, these imperfections are distributed in a complicated fashion along the guide, so that it is difficult to determine the spatial behavior of the coefficients of the fundamental equations for a given fiber, and, also, if they are known, it is practically impossible to deduce an analytical solution.

In order to circumvent these difficulties, it is useful to introduce a statistical ensemble of fibers possessing small random deviations from a common ideal structure [1], [2]. The problem is then to obtain simple equations for the ensemble averages of quantities describing either the evolution of each propagation mode, or the interaction between different modes. The perturbative approach [1], [2] allows one to derive a closed system of equations for the ensemble averages of the powers of the coupled modes, also taking into account losses due to small coupling with radiation modes.[1]

The behavior of the variance of the power has also been investigated [2], in the limit of a large number of coupled modes, thus enabling one to give an estimate of the applicability of the results of the statistical theory to a single fiber.

In this paper, we wish to introduce an analytical approach, which slightly improves the procedure followed in [1] and [2], and allows us to obtain, for a lossless optical fiber, in a straightforward way, beyond the equations for the powers, closed systems of coupled equations for ensemble averages of products of amplitudes of different modes. Furthermore, we obtain a closed system of equations connecting the averages of the power squares to those of the products of different mode powers.

As a particular application, we estimate the normalized variance of the asymptotic power distribution, which turns out to depend on the number of coupled modes.

II. COUPLED POWER EQUATIONS

We start from the relevant deterministic wave equations valid for the single fiber, which couple forward-traveling guided modes and are obtained from the general theory [4] by neglecting coupling with backward-traveling modes and radiation modes. For a steady-state situation, they read

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[1] For good sources on the treatment of stochastic equations, see [3].